Problem 1.33

Test the divergence theorem for the function $\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$. Take as your volume the cube shown in Fig. 1.30, with sides of length 2.

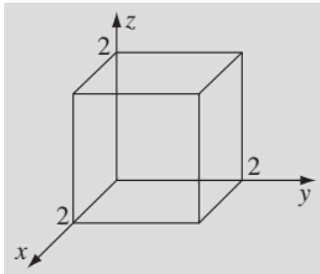


Fig. 1.30

Solution

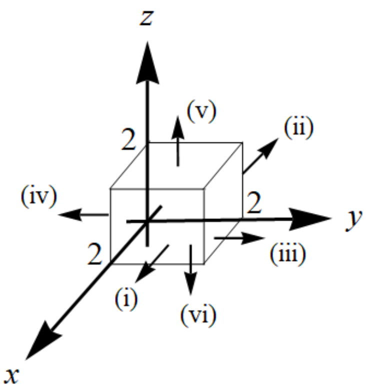
The aim here is to verify Gauss's theorem, which states that

$$\iiint_D \nabla \cdot \mathbf{v}\, dV = \oiint_{\text{bdy } D} \mathbf{v} \cdot d\mathbf{S},$$

for the given cube D. Start by evaluating the volume integral on the left side.

$$\begin{aligned}
\iiint_D \nabla \cdot \mathbf{v}\, dV &= \int_0^2 \int_0^2 \int_0^2 \nabla \cdot \mathbf{v}\, dx\, dy\, dz \\
&= \int_0^2 \int_0^2 \int_0^2 \left[\frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3zx)\right] dx\, dy\, dz \\
&= \int_0^2 \int_0^2 \int_0^2 [(y) + (2z) + (3x)]\, dx\, dy\, dz \\
&= \int_0^2 \int_0^2 \left[(xy) + (2xz) + \left(\frac{3x^2}{2}\right)\right]\Bigg|_0^2 dy\, dz \\
&= \int_0^2 \int_0^2 [(2y) + (4z) + (6)]\, dy\, dz \\
&= \int_0^2 \left[(y^2) + (4yz) + (6y)\right]\Big|_0^2 dz \\
&= \int_0^2 [(4) + (8z) + (12)]\, dz \\
&= [(4z) + (4z^2) + (12z)]\Big|_0^2 \\
&= (8) + (16) + (24) \\
&= 48
\end{aligned}$$

Label the different faces of the cube



and evaluate the closed surface integral on the right side.

$$\oiint_{\text{bdy } D} \mathbf{v}\cdot d\mathbf{S} = \iint_{\text{face (i)}} \mathbf{v}\cdot d\mathbf{S} + \iint_{\text{face (ii)}} \mathbf{v}\cdot d\mathbf{S} + \iint_{\text{face (iii)}} \mathbf{v}\cdot d\mathbf{S} + \iint_{\text{face (iv)}} \mathbf{v}\cdot d\mathbf{S} + \iint_{\text{face (v)}} \mathbf{v}\cdot d\mathbf{S} + \iint_{\text{face (vi)}} \mathbf{v}\cdot d\mathbf{S}$$

$$= \int_0^2\int_0^2 \mathbf{v}\cdot(\hat{\mathbf{x}}\,dy\,dz)\Big|_{x=2} + \int_0^2\int_0^2 \mathbf{v}\cdot(-\hat{\mathbf{x}}\,dy\,dz)\Big|_{x=0} + \int_0^2\int_0^2 \mathbf{v}\cdot(\hat{\mathbf{y}}\,dx\,dz)\Big|_{y=2}$$

$$+ \int_0^2\int_0^2 \mathbf{v}\cdot(-\hat{\mathbf{y}}\,dx\,dz)\Big|_{y=0} + \int_0^2\int_0^2 \mathbf{v}\cdot(\hat{\mathbf{z}}\,dx\,dy)\Big|_{z=2} + \int_0^2\int_0^2 \mathbf{v}\cdot(-\hat{\mathbf{z}}\,dx\,dy)\Big|_{z=0}$$

$$= \int_0^2\int_0^2 v_x\Big|_{x=2} dy\,dz - \int_0^2\int_0^2 v_x\Big|_{x=0} dy\,dz + \int_0^2\int_0^2 v_y\Big|_{y=2} dx\,dz$$

$$- \int_0^2\int_0^2 v_y\Big|_{y=0} dx\,dz + \int_0^2\int_0^2 v_z\Big|_{z=2} dx\,dy - \int_0^2\int_0^2 v_z\Big|_{z=0} dx\,dy$$

$$= \int_0^2\int_0^2 (xy)\Big|_{x=2} dy\,dz - \int_0^2\int_0^2 (xy)\Big|_{x=0} dy\,dz + \int_0^2\int_0^2 (2yz)\Big|_{y=2} dx\,dz$$

$$- \int_0^2\int_0^2 (2yz)\Big|_{y=0} dx\,dz + \int_0^2\int_0^2 (3zx)\Big|_{z=2} dx\,dy - \int_0^2\int_0^2 (3zx)\Big|_{z=0} dx\,dy$$

$$= \int_0^2\int_0^2 (2y)\,dy\,dz - \int_0^2\int_0^2 (0)\,dy\,dz + \int_0^2\int_0^2 (4z)\,dx\,dz$$

$$- \int_0^2\int_0^2 (0)\,dx\,dz + \int_0^2\int_0^2 (6x)\,dx\,dy - \int_0^2\int_0^2 (0)\,dx\,dy$$

$$= 2\int_0^2\int_0^2 y\,dy\,dz + 4\int_0^2\int_0^2 z\,dx\,dz + 6\int_0^2\int_0^2 x\,dx\,dy$$

$$= 2(4) + 4(4) + 6(4)$$

$$= 48$$

Because the left and right sides are equal, Gauss's theorem is verified.